ON LATENCY AND VOLATILITY

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In 1827, Robert Brown used a microscope to discover jittery paths of pollen in water. Modeling these paths, which became known as Brownian motion, laid the foundations for much of modern diffusion theory, which is in turn at the core of many areas of physics and engineering. Brownian motion has also become the primary model for the dynamics of asset prices; just like many water molecules impact pollen particles, trading decisions of many individual investors bombard asset prices and set them on jittery paths.

But what of Brown's microscope? Brown's ability to see the particles (but not the water molecules) was limited by the resolution of his microscope. Similarly , we can observe asset prices, but not latent liquidity and information. Yet, just as microscopy and image technology has advanced to the point where we can almost see and take snapshots of individual atoms (via technologies like atomic force microscopy), modern automated markets can allow for much better *resolution* of the "fundamental" (or "true") price of an asset (much of modern finance, and our work here, is based on the premise of existence of a fundamental but latent price process). One can think of "matching engines" in modern automated trading platforms as financial microscopes transmitting images of the jittery paths of asset prices under different liquidity and informational conditions (akin to water temperature and pollution).

The specific focus of this paper is one aspect of the financial microscope – *latency*. Latency is a delay between an event and its observation. For technological reasons, latency increases with the distance between the matching engine (the microscope) and the market participants (the water molecules). (We note that atomic force microscopy uses not light but *distance* to the specimen.) As such, a race to reduce latency has become a crucial feature of modern automated trading as market participants try to beat each other to get to the asset price as quickly as they they can.

How does latency affect the dynamics of asset prices in modern markets? In this paper, we present a simple model of latency. In our model, latency is a delay between the observed asset price and its true, but latent fundamental price. Because of latency, the observed asset price shadows the true but latent asset price at some deformed distance away. In other words, latency leads to the deformation in the *clock* of an asset's evolution.

Diffusive scaling links time to volatility meaning that deformation in a clock should correspond to fluctuations in volatility; so, latency should be in some way related to the volatility of volatility. A standard way to characterize the volatility of volatility, however, hinges upon first estimating volatility at an intermediate fixed time scale, and then looking at the fluctuations of volatility. This is going to miss the effect of latency. Instead, we define a *volatility of instantaneous volatility* (VIV), which pushes the notion of the volatility of volatility down to a microscopic level and enables us to link volatility to latency. We demonstrate the link between VIV and latency first for a fixed latency and then suggest how to generalize it to *stochastic*, evolving latency.

Intuitively, we show that we can think of latency much as we think of volatility. The true instantaneous volatility is never directly observed, but can still be estimated. For example, computing realized volatility (Section 3) is a way to estimate the "true" but unobserved volatility by summing up appropriate 'microscopic' quantities. Realized volatility scales linearly with time and, upon normalization, leads to an estimate of the true volatility.

We want our understanding of latency to be similar in spirit; it should stem from a sum of microscopic contributions, with the result scaling with time, which upon normalization for the deformation in the clock

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should get us back to the true volatility. In other words, while the evolution of the true price on its native clock is never directly observed, we can show how it can be extracted from realized prices evolving on a deformed clock.

Our model is very simple. We start with the true price evolving as a geometric Brownian motion and take it from there. In reality, of course, prices in modern automated markets are determined by market participants sending messages about their demand schedules to the trading platforms. Trading platforms process messages (with a delay), convert them into orders (with an additional delay), place order into queues (with more delay), and transmit (with yet another delay) information about the changes in the queues to market participants. We roll of this market complexity into a very stylized representation of delay – a notion that by the time market participants get to observe the prices, the true fundamental price has already moved on by some latency increment.

Our bare bones model omits a number of effects in the name of simplicity. Most importantly, we ignore all the complexities inside an automated matching engine and the infrastructure surrounding it; we boldly assume that the matching engine and system architecture around it have been designed to efficiently achieve price discovery. We also neglect the effect of tick sizes and the bid-ask spread. Our interest is in how to think about latency; a better model might combine the effects of latency and the quantization effects enforced by a positive tick size. Our model is a very simplified version of what actually goes on inside modern markets, but we believe that it captures the essence of it.

If you decide to continue reading, you will notice that while the intuition that latency ends up in the volatility of volatility is simple, the math quickly gets out of hand as we need to keep track of many moving parts inside deformed clocks. Some parts move by quite a bit, other move a little, and yet others move by so little that their movement can be ignored. We need to separate all these moving parts, measure their individual movements, and convert it into a deformation scale.

Why go through all this math? Because we believe that with the continuing automation of the trading process, latency itself has become a market factor which should be explicitly modeled. We believe that latency or the distance to the price formation process can interact with changes in market conditions: depending on how "hot" or "polluted" the market is, traders operating at different internal latencies (with which their internal trading systems process changes in market conditions and make trading decisions) can decide to enter or leave the market, making the observed marketwide latency evolve in accordance with some stochastic rule. This means that the price formation process will also evolve in accordance with a stochastic rule that reflects internal latencies of the shifting ecosystem of market participants. Under some circumstances shifts in the ecosystem can give rise to flash crashes or flash rallies as anomalies in latency are detected and acted upon.

If latency and volatility are linked and volatility is traded, why not also trade latency? If a trading strategy is exposed to latency risk (possibility of market prices moving before an internal trading system had a chance to react), those with very low internal latency can sell protection to those with higher internal latency. Traders who wish to speculate on marketwide latency can take positions in something akin to the VIX; let's call the LIX – the Latency Index. This could help price latency risk and spread it around, but in order for it to work, we need to understand and model the fundamentals of latency. With this work we hope to contribute to modeling latency and its effects in modern markets.

Fundamental references to this work are [Clark73],[CIR85], and [ABDL03]. [Clark73] shows that the clock on which asset prices evolve can itself be stochastic or subordinated to a (possibly latent) diffusion process. Our thinking about a stochastic, deformed clock really began with Clark's foreknowledge. Our insight is that the deformation in the clock is due to delays imbedded in the trading technology. [CIR85] propose a strictly positive, mean–reverting diffusion process for the instantaneous interest rate. We believe that this process is suitable for modeling latency because, intuitively, instantaneous interest rate can be thought of as the compensation for being exposed to movement of the fundamental price during the latency period. [ABDL03] show how to estimate instantaneous volatility from transaction prices. This work spawned a long literature that includes recent work on how to correct the estimates of instantaneous volatility for a possibility that prices are observed random intervals apart.

1. Price and Latency

Let's assume that the "fundamental" latent price process S° follows a driftless geometric Brownian motion; i.e.,

$$dS_t^\circ = bS_t^\circ dt + \sigma S_t^\circ dW_t$$
$$S_0^\circ = S_\circ$$

Let's also assume that the observed price process follows

$$S_t^\varepsilon \stackrel{\text{def}}{=} S_{t-\varepsilon\lambda}^\circ.$$

In other words, S_t^ε shadows the fundamental price process S° with a delay of $\varepsilon\lambda$. For now, let's fix $\lambda > 0$; in Section 5 we will let λ evolve dynamically. Let's also fix a small scaling parameter $\varepsilon \geq 0$. The parameter ε should be thought of as a technological constant which gives the speed at which price discovery takes place. Currently, in well–optimized automated platforms, $\varepsilon \approx 10^{-6}$. By letting ε approach zero, we will consider the effect of latency on the observed price process. At the limit, of course, when $S_t^\varepsilon = S_t^0$ by taking $\varepsilon = 0$ in S^ε, the prices S_t^0 and S_t° (the fundamental price) coincide; we perfectly observe the fundamental price if there is no latency.

Our goal is to characterize the behavior of S^ε (compared to S°) for $\varepsilon \ll 1$. Namely, we would like to characterize the effect of latency when we try to calculate volatility from the delayed prices.

2. Sampling Times

The next aspect of the model is the times at which prices (fundamental or latent) are observed. For simplicity, we will focus on transaction prices rather than all the prices in the queues and say that executions occur at multiples of some (small) parameter $\varkappa$. Define

$$\tau_n^\varkappa \stackrel{\text{def}}{=} \varkappa n$$

for $n \in \{0, 1 \dots\}$. We will eventually let $\varkappa \searrow 0$, so that the intertrade duration times (times between consecutive executions) become small; i.e., executions become more and more frequent. Let's also now define

$$\tau_n^{\varepsilon,\varkappa} \stackrel{\text{def}}{=} \tau_n^\varkappa - \varepsilon\lambda = n\varkappa - \varepsilon\lambda;$$

although $\tau_n^\varkappa$ is the actual time of an execution, the price seen at that time is $S_{\tau_n^\varkappa}^\varepsilon = S_{\tau_n^{\varepsilon,\varkappa}}^\circ$, i.e., it is the "fundamental" price at time $\tau_n^{\varepsilon,\varkappa}$ delayed by $\varepsilon\lambda$. Note that since $\tau_{n+1}^\varkappa > \tau_n^\varkappa$, we also have that $\tau_{n+1}^{\varepsilon,\varkappa} > \tau_n^{\varepsilon,\varkappa}$. See Figure 1.

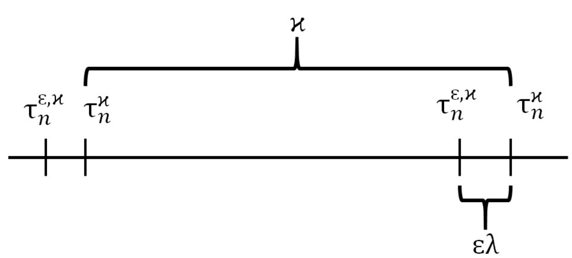

Figure 1. Sampling Times: Actual and Observed With a Delay

We want look at a regime where ε and $\varkappa$ simultaneously become small. However, we want to do so in a way such that

$$\frac{\varepsilon}{\varkappa} \searrow 0.$$

In other words, latency is smaller than the intertrade duration time. Of course, if $\varepsilon \gg \varkappa$, i.e., the delay in observing prices is much larger than intergrade duration, then for some reason available trading technology (parametrized by ε) is not adequate for the underlying price discovery process.

To get to executions that take place by some distant time, define for positive T and $\varkappa$,

$$\mathsf{N}^{\varkappa}(T) \stackrel{\text{def}}{=} \sup\{n \in \mathbb{N} : \tau_n^{\varkappa} \le T\} = \left\lfloor \frac{t}{\varkappa} \right\rfloor.$$

Then,

$$\left|\tau^{\varkappa}_{\mathsf{N}^{\varkappa}(T)} - T\right| \le \varkappa \qquad \text{and} \qquad \left|\tau^{\varkappa,\varepsilon}_{\mathsf{N}^{\varkappa}(T)} - T\right| \le \varkappa + \varepsilon\lambda.$$

3. Latency, Volatility, and the Volatility of Volatility

To efficiently focus on volatility, let's define

$$Z_t \stackrel{\text{def}}{=} \ln S_t;$$

i.e., we will consider *log returns.* We have that

$$Z_t = \ln S_\circ + \sigma W_t + \left(b - \tfrac{1}{2}\sigma^2\right) t.$$

To proceed, let's first recall how to compute the realized volatility of the fundamental price process S°; i.e., let's assume complete observation of S° at the instants of execution. Define

$$\boldsymbol{\sigma}^{\varkappa}(T) \stackrel{\text{def}}{=} \sum_{1 \le n \le \mathsf{N}^{\varkappa}(T)} \left(\ln \frac{S^\circ_{\tau^{\varkappa}_n}}{S^\circ_{\tau^{\varkappa}_{n-1}}}\right)^2 = \sum_{1 \le n \le \mathsf{N}^{\varkappa}(T)} \left(Z_{\tau^{\varkappa}_n} - Z_{\tau^{\varkappa}_{n-1}}\right)^2.$$

For $\varkappa \ll 1$, the dominant behavior of the right-hand side is given by

$$\left(Z_{\tau^{\varkappa}_n} - Z_{\tau^{\varkappa}_{n-1}}\right)^2 \approx \sigma^2(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1}); \tag{1}$$

hence

$$\boldsymbol{\sigma}^{\varkappa}(T) \approx \sum_{1 \le n \le \mathsf{N}^{\varkappa}(T)} \sigma^2\{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1}\} = \sigma^2 \tau^{\varkappa}_{\mathsf{N}^{\varkappa}(T)} \approx \sigma^2 T. \tag{2}$$

The same calculation holds when latency is introduced; the effect of latency telescopes, i.e., only the first delay and last delay matter during the period between the first and the N's transaction, and so is negligible. Quantitatively, let's define

$$\boldsymbol{\sigma}^{\varepsilon,\varkappa}(T) \stackrel{\text{def}}{=} \sum_{1 \le n \le \mathsf{N}^{\varkappa}(T)} \left(\ln \frac{S^\varepsilon_{\tau^{\varkappa}_n}}{S^\varepsilon_{\tau^{\varkappa}_{n-1}}}\right)^2.$$

If $\varepsilon \ll 1$ and $\varkappa \ll 1$,

$$\begin{aligned}\boldsymbol{\sigma}^{\varepsilon,\varkappa}(T) &= \sum_{1 \le n \le \mathsf{N}^{\varkappa}(T)} \left(Z_{\tau^{\varepsilon,\varkappa}_n} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}}\right)^2 \approx \sigma^2 \sum_{1 \le n \le \mathsf{N}^{\varkappa}(T)} \left(\tau^{\varepsilon,\varkappa}_n - \tau^{\varepsilon,\varkappa}_{n-1}\right) \\ &= \sigma^2 \tau^{\varepsilon,\varkappa}_{\mathsf{N}^{\varkappa}(T)} \approx \sigma^2 T\end{aligned} \tag{3}$$

Intuitively, addition of λ should be interpreted as deformations in the *clock.* The point of the above calculation is that these deformations tend to 'average out' in the same way that a telescoping series can be reduced to endpoint behavior. To capture deformations in the clock, we should calculate the *volatility of volatility* – fluctuations of volatility due to delays within the clock.

Instead of first estimating volatility over a number of returns at an intermediate fixed time scale and then looking at the fluctuations of this estimated volatility, let's look at this microscopically. If we have transactions at times t and t' with $t' > t$, we can estimate the volatility between t and t' as

$$\frac{\left(\ln \frac{S^\circ_{t'}}{S^\circ_t}\right)^2}{t' - t} = \frac{(Z_{t'} - Z_t)^2}{t' - t} \tag{4}$$

Define the *volatility of instantaneous volatility* (VIV) as

$$\mathbf{VIV}^{\varkappa}(T) \stackrel{\text{def}}{=} \sum_{1\le n<\mathsf{N}^{\varkappa}(T)} \left\{ \frac{\left(\ln \frac{S^{\circ}_{\tau^{\varkappa}_{n+1}}}{S^{\circ}_{\tau^{\varkappa}_{n}}}\right)^2}{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n}} - \frac{\left(\ln \frac{S^{\circ}_{\tau^{\varkappa}_{n}}}{S^{\circ}_{\tau^{\varkappa}_{n-1}}}\right)^2}{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1}} \right\}^2 \left\{ \frac{\tau^{\varkappa}_{n+1}+\tau^{\varkappa}_{n}}{2} - \frac{\tau^{\varkappa}_{n}+\tau^{\varkappa}_{n-1}}{2} \right\}$$

$$= \frac{1}{2} \sum_{1\le n<\mathsf{N}^{\varkappa}(T)} \left\{ \frac{\left(Z_{\tau^{\varkappa}_{n+1}} - Z_{\tau^{\varkappa}_{n}}\right)^2}{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n}} - \frac{\left(Z_{\tau^{\varkappa}_{n}} - Z_{\tau^{\varkappa}_{n-1}}\right)^2}{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1}} \right\}^2 \left\{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}\right\}$$

This quantifies the fluctuations of the microscopic estimates of volatility. As we shall see, scaling by the difference in time gives us a quantity which grows linearly (like (2) and (3)).

Let's begin with a Gaussian calculation. For each $n\in\mathbb{N}$ and $\varkappa>0$, define

$$\eta^{\varkappa}_n \stackrel{\text{def}}{=} \frac{W_{\tau^{\varkappa}_n} - W_{\tau^{\varkappa}_{n-1}}}{\sqrt{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1}}};$$

for each $\varkappa$, $\{\eta^{\varkappa}_n\}_{n\in\mathbb{N}}$ is an i.i.d. collection of standard Gaussians (we use here the fact that the $\tau^{\varkappa}_n$'s are independent of W). From (4), we then have that

$$\frac{\left(Z_{\tau^{\varkappa}_n} - Z_{\tau^{\varkappa}_{n-1}}\right)^2}{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1}} \approx \sigma^2 (\eta^{\varkappa}_n)^2;$$

at small scales, diffusion terms dominate drift terms. Thus

$$\mathbf{VIV}^{\varkappa}(T) \approx \frac{\sigma^4}{2} \sum_{1\le n<\mathsf{N}^{\varkappa}(T)} \left\{ (\eta^{\varkappa}_n)^2 - \left(\eta^{\varkappa}_{n-1}\right)^2 \right\}^2 (\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1})$$

$$= \frac{\sigma^4}{2} \sum_{1\le n<\mathsf{N}^{\varkappa}(T)} \left\{ \left(\eta^{\varkappa}_{n+1}\right)^4 + (\eta^{\varkappa}_n)^4 - 2\left(\eta^{\varkappa}_{n+1}\right)^2 (\eta^{\varkappa}_n)^2 \right\} (\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1})$$

If η is a standard Gaussian, then

(5) $$\mathbb{E}[\eta^4]=3 \qquad \text{and} \qquad \mathbb{E}[\eta^2]=1.$$

A fairly simple modification of the law of large numbers then implies that

$$\mathbf{VIV}^{\varkappa}(T) \approx \frac{\sigma^4}{2} \sum_{1\le n<\mathsf{N}^{\varkappa}(T)} \{3+3-2\} \left\{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}\right\} \approx \frac{\sigma^4}{2}\times 4\times (2T) = 4\sigma^4 T.$$

4. Latency and VIV

Let's see how we can recover the volatility of an asset in the presence of latency. Define

$$\mathbf{VIV}^{\varepsilon,\varkappa}(T) \stackrel{\text{def}}{=} \sum_{1\le n<\mathsf{N}^{\varkappa}(T)} \left\{ \frac{\left(\ln \frac{S^{\varepsilon}_{\tau^{\varkappa}_{n+1}}}{S^{\varepsilon}_{\tau^{\varkappa}_{n}}}\right)^2}{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n}} - \frac{\left(\ln \frac{S^{\varepsilon}_{\tau^{\varkappa}_{n}}}{S^{\varepsilon}_{\tau^{\varkappa}_{n+1}}}\right)^2}{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1}} \right\}^2 \left\{ \frac{\tau^{\varkappa}_{n+1}+\tau^{\varkappa}_{n}}{2} - \frac{\tau^{\varkappa}_{n}+\tau^{\varkappa}_{n-1}}{2} \right\}$$

$$= \frac{1}{2} \sum_{1\le n<\mathsf{N}^{\varepsilon,\varkappa}(T)} \left\{ \frac{\left(\ln \frac{S^{\circ}_{\tau^{\varepsilon,\varkappa}_{n+1}}}{S^{\circ}_{\tau^{\varepsilon,\varkappa}_{n}}}\right)^2}{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n}} - \frac{\left(\ln \frac{S^{\circ}_{\tau^{\varepsilon,\varkappa}_{n}}}{S^{\circ}_{\tau^{\varepsilon,\varkappa}_{n-1}}}\right)^2}{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1}} \right\}^2 \left\{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}\right\}$$

$$= \frac{1}{2} \sum_{1\le n<\mathsf{N}^{\varepsilon,\varkappa}(T)} \left\{ \frac{\left(Z_{\tau^{\varepsilon,\varkappa}_{n+1}} - Z_{\tau^{\varepsilon,\varkappa}_{n}}\right)^2}{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n}} - \frac{\left(Z_{\tau^{\varepsilon,\varkappa}_{n}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}}\right)^2}{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1}} \right\}^2 \left\{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}\right\}$$

Our main result is a comparison of $\mathbf{VIV}^{\varepsilon,\varkappa}(T)$ and $\mathbf{VIV}^{\varkappa}(T)$; more precisely it is an asymptotic approximation of $\mathbf{VIV}^{\varepsilon,\varkappa} - \mathbf{VIV}^{\varkappa}$.

Theorem 4.1. *We have that*

$$\boldsymbol{VIV}^{\varepsilon,\varkappa}(T) \approx \boldsymbol{VIV}^{\varkappa}(T) + \varepsilon M_T$$

where M is a martingale with quadratic variation

$$\langle M\rangle_T = C_{(8)}\sigma^4\lambda T$$

where $C_{(8)}$ is the constant given in (8)*. In other words, if we define*

$$M_t^{\varepsilon,\varkappa} \overset{def}{=} \varepsilon^{-1}\left\{\boldsymbol{VIV}^{\varepsilon,\varkappa}(t) - \boldsymbol{VIV}^{\varkappa}(t)\right\}$$

then

$$M \overset{def}{=} \lim_{\substack{\varepsilon,\varkappa\searrow 0\\ \varepsilon/\varkappa\searrow 0}} M_t^{\varepsilon,\varkappa}$$

is a martingale with quadratic variation $\langle M\rangle_t = C_{(8)}\sigma^4\lambda T$.

Theorem 4.1 is an asymptotic result; volatility or the average value of $\mathbf{VIV}^{\varepsilon,\varkappa}$ agrees with that of $\mathbf{VIV}^{\varkappa}$ but the variance of $\mathbf{VIV}^{\varepsilon,\varkappa}$ grows faster than that of $\mathbf{VIV}^{\varkappa}$, at a rate which is proportional to the latency λ.

4.1. **Setup.** We want to prove Theorem 4.1 by using approximations like (1) and (5); more generally, we have that

$$(Z_t - Z_s)^2 \approx \sigma^2(t-s), \qquad (Z_t - Z_s)^4 \approx 3\sigma^4(t-s)^2, \qquad \text{and} \qquad (Z_t - Z_s)^6 \approx 15\sigma^6(t-s)^3 \tag{6}$$

if $0 \le t - s \ll 1$.

Let's start with some representation results. If $0 \le r' < r < s' < s < t' < t$, define

$$\begin{aligned}
\Xi_{r',r,s',s,t',t} &= \left\{\frac{(Z_t - Z_s)^2}{t-s} - \frac{(Z_s - Z_r)^2}{s-r}\right\}^2 - \left\{\frac{(Z_{t'} - Z_{s'})^2}{t-s} - \frac{(Z_{s'} - Z_{r'})^2}{s-r}\right\}^2\\
&= \frac{(Z_t - Z_s)^4 - (Z_{t'} - Z_{s'})^4}{(t-s)^2}\\
&\quad - 2\frac{(Z_t - Z_s)^2(Z_s - Z_r)^2 - (Z_{t'} - Z_{s'})^2(Z_{s'} - Z_{r'})^2}{(t-s)(s-r)}\\
&\quad + \frac{(Z_s - Z_r)^4 - (Z_{s'} - Z_{r'})^4}{(s-r)^2}
\end{aligned}$$

As a useful tool, note the formula

$$\begin{aligned}
W_t^3 &= 3\int_{r=0}^t W_r^2 dW_r + 3\int_{r=0}^t W_r dr\\
&= 3\int_{r=0}^t W_r^2 dW_r + 3\int_{r=0}^t (t-r) dW_r;
\end{aligned}$$

this gives a formula for W_t^3 purely in terms of a dW integral.

Let's identify some relevant asymptotics. If $0 \le a' < a < b' < b$ and $b - b' \ll b' - a$ and $a - a' \ll b' - a$, we expect that $|Z_b - Z_{b'}| \ll |Z_{b'} - Z_a|$ and $|Z_a - Z_{a'}| \ll |Z_{b'} - Z_a'|$. Then

$$\begin{aligned}
&(Z_b - Z_a)^4 - (Z_{b'} - Z_{a'})^4\\
&\quad = \{(Z_b - Z_a)^4 - (Z_{b'} - Z_a)^4\} - \{(Z_{b'} - Z_{a'})^4 - (Z_{b'} - Z_a)^4\}\\
&\quad = \{(Z_b - Z_{b'} + Z_{b'} - Z_a)^4 - (Z_{b'} - Z_a)^4\} - \{(Z_{b'} - Z_a + Z_a - Z_{a'})^4 - (Z_{b'} - Z_a)^4\}\\
&\quad \approx 4(Z_{b'} - Z_a)^3(Z_b - Z_{b'}) - 4(Z_{b'} - Z_a)^3(Z_a - Z_{a'})\\
&\quad = 4\int_{u=b'}^b (Z_{b'} - Z_a)^3 dZ_u - 4\left\{3\int_{u=a}^{b'} (Z_u - Z_a)^2 dZ_u + 3\int_{u=a}^{b'} (b' - u) dZ_u\right\}(Z_a - Z_{a'})
\end{aligned}$$

$$= 4\left\{\int_{u=b'}^{b}(Z_{b'}-Z_a)^3 dZ_u - 3\int_{u=a}^{b'}(Z_u-Z_a)^2(Z_a-Z_{a'})dZ_u - 3\int_{u=a}^{b'}(b'-u)(Z_a-Z_{a'})dZ_u\right\}.$$

Also, if $0 \le r' \le r \le s' \le s \le t' \le t$ and $t-t'$, $s-s'$, and $r-r'$ are small compared to $t'-s$ and $s'-r$,

$$\begin{aligned}
&(Z_t-Z_s)^2(Z_s-Z_r)^2 - (Z_{t'}-Z_{s'})^2(Z_{s'}-Z_{r'})^2 \\
&\quad = (Z_t-Z_s)^2\{(Z_s-Z_r)^2-(Z_{s'}-Z_{r'})^2\} + \{(Z_t-Z_s)^2-(Z_{t'}-Z_{s'})^2\}(Z_{s'}-Z_{r'})^2 \\
&\quad = (Z_t-Z_s)^2\left\{\{(Z_s-Z_r)^2-(Z_{s'}-Z_r)^2\}+\left\{(Z_{s'}-Z_r)^2-(Z_{s'}-Z_{r'})^2\right\}\right\} \\
&\qquad + \left\{\{(Z_t-Z_s)^2-(Z_{t'}-Z_s)^2\}+\{(Z_{t'}-Z_s)^2-(Z_{t'}-Z_{s'})^2\}\right\}(Z_{s'}-Z_{r'})^2 \\
&\quad = (Z_t-Z_s)^2\left\{\{(Z_s-Z_{s'}+Z_{s'}-Z_r)^2-(Z_{s'}-Z_r)^2\}\right. \\
&\qquad\qquad \left.+\{(Z_{s'}-Z_r)^2-(Z_{s'}-Z_r+Z_r-Z_{r'})^2\}\right\} \\
&\qquad + \left\{\{(Z_t-Z_{t'}+Z_{t'}-Z_s)^2-(Z_{t'}-Z_s)^2\}\right. \\
&\qquad\qquad \left.+\{(Z_{t'}-Z_s)^2-(Z_{t'}-Z_s+Z_s-Z_{s'})^2\}\right\}(Z_{s'}-Z_{r'})^2 \\
&\quad \approx (Z_t-Z_s)^2\{2(Z_s-Z_{s'})(Z_{s'}-Z_r)-2(Z_{s'}-Z_r)(Z_r-Z_{r'})\} \\
&\qquad + \{2(Z_t-Z_{t'})(Z_{t'}-Z_s)-2(Z_{t'}-Z_s)(Z_s-Z_{s'})\}(Z_{s'}-Z_{r'})^2 \\
&\quad = \left\{2\int_{u=s}^{t}(Z_u-Z_s)dZ_u+(t-s)\right\}\{2(Z_s-Z_{s'})(Z_{s'}-Z_r)-2(Z_{s'}-Z_r)(Z_r-Z_{r'})\} \\
&\qquad + \left\{2\int_{u=t'}^{t}dZ_u(Z_{t'}-Z_s)-2\int_{u=s}^{t'}dZ_u(Z_s-Z_{s'})\right\}(Z_{s'}-Z_{r'})^2 \\
&\quad = 4\int_{u=s}^{t}(Z_u-Z_s)\{(Z_s-Z_{s'})(Z_{s'}-Z_r)-(Z_{s'}-Z_r)(Z_r-Z_{r'})\}dZ_u \\
&\qquad + 2\int_{u=s'}^{s}(t-s)(Z_{s'}-Z_r)dZ_u - 2\int_{u=r}^{s'}(t-s)(Z_r-Z_{r'})dZ_u \\
&\qquad + 2\int_{u=t'}^{t}(Z_{t'}-Z_s)(Z_{s'}-Z_{r'})^2dZ_u - 2\int_{u=s}^{t'}(Z_s-Z_{s'})(Z_{s'}-Z_{r'})^2dZ_u
\end{aligned}$$

Using these formulae, we have that

$$\begin{aligned}
\Xi_{r',r,s',s,t',t}(t-r) = {}& 4\left\{\int_{u=t'}^{t}(Z_{t'}-Z_s)^3dZ_u - 3\int_{u=s}^{t'}(Z_u-Z_s)^2(Z_s-Z_{s'})dZ_u\right. \\
&\qquad \left. -3\int_{u=s}^{t'}(t'-u)(Z_s-Z_{s'})dZ_u\right\}\frac{t-r}{(t-s)^2} \\
& + 4\left\{\int_{u=s'}^{s}(Z_{s'}-Z_r)^3dZ_u - 3\int_{u=r}^{s'}(Z_u-Z_r)^2(Z_r-Z_{r'})dZ_u\right. \\
&\qquad \left. -3\int_{u=r}^{s'}(s'-u)(Z_r-Z_{r'})dZ_u\right\}\frac{t-r}{(s-r)^2} \\
& - 8\left\{\int_{u=s}^{t}(Z_u-Z_s)\{(Z_s-Z_{s'})(Z_{s'}-Z_r)-(Z_{s'}-Z_r)(Z_r-Z_{r'})\}dZ_u\right\}\frac{t-r}{(t-s)(s-r)} \\
& - 4\left\{\int_{u=s'}^{s}(t-s)(Z_{s'}-Z_r)dZ_u - \int_{u=r}^{s'}(t-s)(Z_r-Z_{r'})dZ_u\right\}\frac{t-r}{(t-s)(s-r)} \\
& - 4\left\{\int_{u=t'}^{t}(Z_{t'}-Z_s)(Z_{s'}-Z_{r'})^2dZ_u - \int_{u=s}^{t'}(Z_s-Z_{s'})(Z_{s'}-Z_{r'})^2dZ_u\right\}\frac{t-r}{(t-s)(s-r)}
\end{aligned}$$

4.2. **Martingale Representation.** We thus have a *martingale approximation*;

$$\mathbf{VIV}^{\varepsilon,\varkappa}(T) - \mathbf{VIV}^{\varkappa}(T) = 2 \sum_{\mathsf{N}^{\varepsilon,\varkappa}(0)<n<\mathsf{N}^{\varepsilon,\varkappa}(T)} \left\{ \int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_n^{\varepsilon,\varkappa}(u) dZ_u \right\} \tag{7}$$

where

$$\begin{aligned}
v_n^{\varepsilon,\varkappa}(u) = \Bigg\{ & (Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}})^3 \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} + (Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}})^3 \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \\
& -(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}})(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^2 \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \\
& -(\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}}) \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})} \Bigg\} \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^{\varkappa}]}(u) \\
- \Bigg\{ & 3\{(Z_u - Z_{\tau_{n-1}^{\varkappa}})^2 + (\tau_n^{\varepsilon,\varkappa} - u)\}(Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}}) \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \\
& +3\{(Z_u - Z_{\tau_{n-1}^{\varkappa}})^2 + (\tau_n^{\varepsilon,\varkappa} - u)\}(Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}}) \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \\
& -(\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}}) \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})} \\
& -(Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^2 \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \Bigg\} \chi_{(\tau_{n-1}^{\varkappa}, \tau_n^{\varepsilon,\varkappa}]}(u) \\
- 2(Z_u - Z_{\tau_{n-1}^{\varkappa}}) \Big\{ & (Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}}) \\
& -(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}})(Z_{\tau_{n-2}^{\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}}) \Big\} \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \\
= \sum_{j=1}^{9} & v_{n,j}^{\varepsilon,\varkappa}(u)
\end{aligned}$$

where

$$\begin{aligned}
v_{n,1}^{\varepsilon,\varkappa}(u) &\overset{\text{def}}{=} (Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}})^3 \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} + \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \right\} \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^{\varkappa}]}(u) \\
v_{n,2}^{\varepsilon,\varkappa}(u) &\overset{\text{def}}{=} -(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}})(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^2 \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^{\varkappa}]}(u) \\
v_{n,3}^{\varepsilon,\varkappa}(u) &\overset{\text{def}}{=} -(\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}}) \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})} \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^{\varkappa}]}(u) \\
v_{n,4}^{\varepsilon,\varkappa}(u) &\overset{\text{def}}{=} -3(Z_u - Z_{\tau_{n-1}^{\varkappa}})^2 (Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}}) \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} + \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \right\} \chi_{(\tau_{n-1}^{\varkappa}, \tau_n^{\varepsilon,\varkappa}]}(u) \\
v_{n,5}^{\varepsilon,\varkappa}(u) &\overset{\text{def}}{=} -3(\tau_n^{\varepsilon,\varkappa} - u)(Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}}) \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} + \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \right\} \chi_{(\tau_{n-1}^{\varkappa}, \tau_n^{\varepsilon,\varkappa}]}(u) \\
v_{n,6}^{\varepsilon,\varkappa}(u) &\overset{\text{def}}{=} (\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}}) \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})} \chi_{(\tau_{n-1}^{\varkappa}, \tau_n^{\varepsilon,\varkappa}]}(u) \\
v_{n,7}^{\varepsilon,\varkappa}(u) &\overset{\text{def}}{=} (Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^2 \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \chi_{(\tau_{n-1}^{\varkappa}, \tau_n^{\varepsilon,\varkappa}]}(u) \\
v_{n,8}^{\varepsilon,\varkappa}(u) &\overset{\text{def}}{=} -2(Z_u - Z_{\tau_{n-1}^{\varkappa}})(Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}}) \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \\
v_{n,9}^{\varepsilon,\varkappa}(u) &\overset{\text{def}}{=} 2(Z_u - Z_{\tau_{n-1}^{\varkappa}})(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}})(Z_{\tau_{n-2}^{\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}}) \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})}
\end{aligned}$$

Using these expressions, we want to compute the quadratic variation of the right-hand side of (7); we want to evaluate

$$A_t^{\varepsilon,\varkappa} \stackrel{\text{def}}{=} \sum_{\mathsf{N}^{\varepsilon,\varkappa}(0)<n<\mathsf{N}^{\varepsilon,\varkappa}(T)} \int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} (v_n^{\varepsilon,\varkappa}(u))^2 du = \sum_{1\le j,j'\le 9} A_t^{\varepsilon,\varkappa,j,j'}$$

where

$$A_t^{\varepsilon,\varkappa,j,j'} = \sum_{\mathsf{N}^{\varepsilon,\varkappa}(0)<n<\mathsf{N}^{\varepsilon,\varkappa}(T)} \int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,j}^{\varepsilon,\varkappa}(u) v_{n,j'}^{\varepsilon,\varkappa}(u) du.$$

Let's gradually work our way through the different terms. Note that $v_{n,j}^{\varepsilon,\varkappa}$ has support on $(\tau_n^{\varepsilon,\varkappa}, \tau_n^{\varkappa}]$ for $j\in\{1,2,3\}$ and $v_{n,j}^{\varepsilon,\varkappa}$ has support on $(\tau_{n-1}^{\varkappa,\varepsilon}, \tau_n^{\varkappa}]$ for $j\in\{4,5,6,7\}$, and $v_{n,j}^{\varepsilon,\varkappa}$ has support on all of $(\tau_{n-1}^{\varkappa}, \tau_n^{\varkappa}]$ for $j\in\{8,9\}$. As a consequence,

$$v_{n,j}^{\varepsilon,\varkappa}(u) v_{n,j'}^{\varepsilon,\varkappa}(u) = 0$$

for $j\in\{1,2,3\}$ and $j'\in\{4,5,6,7\}$.

Let's write

$$\begin{aligned} A_t^{\varepsilon,\varkappa} &= \sum_{1\le j\le 3} A_t^{\varepsilon,\varkappa,j,j} + 2\sum_{1\le j<j'\le 3} A_t^{\varepsilon,\varkappa,j,j} + \sum_{4\le j\le 7} A_t^{\varepsilon,\varkappa,j,j} + 2\sum_{4\le j<j'\le 7} A_t^{\varepsilon,\varkappa,j,j} \\ &\quad + \sum_{8\le j\le 9} A_t^{\varepsilon,\varkappa,j,j} + 2A_t^{\varepsilon,\varkappa,8,9} + 2\sum_{\substack{1\le j\le 3\\ 8\le j'\le 9}} A_t^{\varepsilon,\varkappa,j,j} + 2\sum_{\substack{4\le j\le 7\\ 8\le j'\le 9}} A_t^{\varepsilon,\varkappa,j,j}. \end{aligned}$$

4.3. **Simplifications:** (6)**.** Let's proceed by using (6). Let's start with the diagonal terms. We directly see that

$$\begin{aligned} (v_{n,1}^{\varepsilon,\varkappa}(u))^2 &= (Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}})^6 \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} + \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \right\}^2 \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^{\varkappa}]}(u) \\ (v_{n,2}^{\varepsilon,\varkappa}(u))^2 &= (Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}})^2 (Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^4 \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \right\}^2 \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^{\varkappa}]}(u) \\ (v_{n,3}^{\varepsilon,\varkappa}(u))^2 &= (\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})^2 (Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^{\varkappa}})^2 \left\{ \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})} \right\}^2 \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^{\varkappa}]}(u) \end{aligned}$$

and

$$\begin{aligned} (v_{n,4}^{\varepsilon,\varkappa}(u))^2 &= 9(Z_u - Z_{\tau_{n-1}^{\varkappa}})^4 (Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^2 \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} + \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \right\}^2 \chi_{(\tau_{n-1}^{\varkappa}, \tau_n^{\varepsilon,\varkappa}]}(u) \\ (v_{n,5}^{\varepsilon,\varkappa}(u))^2 &= 9(\tau_n^{\varepsilon,\varkappa} - u)^2 (Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^2 \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} + \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \right\}^2 \chi_{(\tau_{n-1}^{\varkappa}, \tau_n^{\varepsilon,\varkappa}]}(u) \\ (v_{n,6}^{\varepsilon,\varkappa}(u))^2 &= (\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})^2 (Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^2 \left\{ \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_{n+1}^{\varkappa} - \tau_n^{\varkappa})(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})} \right\}^2 \chi_{(\tau_{n-1}^{\varkappa}, \tau_n^{\varepsilon,\varkappa}]}(u) \\ (v_{n,7}^{\varepsilon,\varkappa}(u))^2 &= (Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^2 (Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^4 \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \right\}^2 \chi_{(\tau_{n-1}^{\varkappa}, \tau_n^{\varepsilon,\varkappa}]}(u) \end{aligned}$$

and

$$\begin{aligned} (v_{n,8}^{\varepsilon,\varkappa}(u))^2 &= 4(Z_u - Z_{\tau_{n-1}^{\varkappa}})^2 (Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^2 (Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}})^2 \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \right\}^2 \\ (v_{n,9}^{\varepsilon,\varkappa}(u))^2 &= 4(Z_u - Z_{\tau_{n-1}^{\varkappa}})^2 (Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}})^2 (Z_{\tau_{n-2}^{\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^2 \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})(\tau_{n-1}^{\varkappa} - \tau_{n-2}^{\varkappa})} \right\}^2 \end{aligned}$$

Using (6), we consequently have

$$(v_{n,1}^{\varepsilon,\varkappa}(u))^2 \approx 15(\tau_n^{\varepsilon,\varkappa} - \tau_{n-1}^{\varkappa})^3 \left\{ \frac{\tau_n^{\varkappa} - \tau_{n-2}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} + \frac{\tau_{n+1}^{\varkappa} - \tau_{n-1}^{\varkappa}}{(\tau_n^{\varkappa} - \tau_{n-1}^{\varkappa})^2} \right\}^2 \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^{\varkappa}]}(u)$$

$$(v^{\varepsilon,\varkappa}_{n,2}(u))^2 \approx 3(\tau^{\varepsilon,\varkappa}_n - \tau^{\varkappa}_{n-1})(\tau^{\varepsilon,\varkappa}_{n-1} - \tau^{\varepsilon,\varkappa}_{n-2})^2 \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1} - \tau^{\varkappa}_{n-2})} \right\}^2 \chi_{(\tau^{\varepsilon,\varkappa}_n, \tau^{\varkappa}_n]}(u)$$

$$(v^{\varepsilon,\varkappa}_{n,3}(u))^2 \approx (\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)^2(\tau^{\varepsilon,\varkappa}_n - \tau^{\varkappa}_{n-1}) \left\{ \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})} \right\}^2 \chi_{(\tau^{\varepsilon,\varkappa}_n, \tau^{\varkappa}_n]}(u)$$

and

$$(v^{\varepsilon,\varkappa}_{n,4}(u))^2 \approx 27(u - \tau^{\varkappa}_{n-1})^2(\tau^{\varkappa}_{n-1} - \tau^{\varepsilon,\varkappa}_{n-1}) \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} \right\}^2 \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varepsilon,\varkappa}_n]}(u)$$

$$(v^{\varepsilon,\varkappa}_{n,5}(u))^2 \approx 9(\tau^{\varepsilon,\varkappa}_n - u)^2(\tau^{\varkappa}_{n-1} - \tau^{\varepsilon,\varkappa}_{n-1}) \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} \right\}^2 \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varepsilon,\varkappa}_n]}(u)$$

$$(v^{\varepsilon,\varkappa}_{n,6}(u))^2 \approx (\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)^2(\tau^{\varkappa}_{n-1} - \tau^{\varepsilon,\varkappa}_{n-1}) \left\{ \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})} \right\}^2 \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varepsilon,\varkappa}_n]}(u)$$

$$(v^{\varepsilon,\varkappa}_{n,7}(u))^2 \approx 3(\tau^{\varkappa}_{n-1} - \tau^{\varepsilon,\varkappa}_{n-1})(\tau^{\varepsilon,\varkappa}_{n-1} - \tau^{\varepsilon,\varkappa}_{n-2})^2 \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1} - \tau^{\varkappa}_{n-2})} \right\}^2 \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varepsilon,\varkappa}_n]}(u)$$

and

$$(v^{\varepsilon,\varkappa}_{n,8}(u))^2 \approx 4(u - \tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1} - \tau^{\varepsilon,\varkappa}_{n-1})(\tau^{\varepsilon,\varkappa}_{n-1} - \tau^{\varkappa}_{n-2}) \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1} - \tau^{\varkappa}_{n-2})} \right\}^2$$

$$(v^{\varepsilon,\varkappa}_{n,9}(u))^2 \approx 4(u - \tau^{\varkappa}_{n-1})(\tau^{\varepsilon,\varkappa}_{n-1} - \tau^{\varkappa}_{n-2})(\tau^{\varkappa}_{n-2} - \tau^{\varepsilon,\varkappa}_{n-2}) \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1} - \tau^{\varkappa}_{n-2})} \right\}^2$$

Let's next look at cross-terms. We have that

$$\begin{aligned}
v^{\varepsilon,\varkappa}_{n,1}(u)v^{\varepsilon,\varkappa}_{n,2}(u) &= -(Z_{\tau^{\varepsilon,\varkappa}_n} - Z_{\tau^{\varkappa}_{n-1}})^4(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&\quad \times \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1} - \tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varepsilon,\varkappa}_n, \tau^{\varkappa}_n]}(u) \\
v^{\varepsilon,\varkappa}_{n,1}(u)v^{\varepsilon,\varkappa}_{n,3}(u) &= -(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)(Z_{\tau^{\varepsilon,\varkappa}_n} - Z_{\tau^{\varkappa}_{n-1}})^4 \\
&\quad \times \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})} \right\} \chi_{(\tau^{\varepsilon,\varkappa}_n, \tau^{\varkappa}_n]}(u) \\
v^{\varepsilon,\varkappa}_{n,2}(u)v^{\varepsilon,\varkappa}_{n,3}(u) &= (\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)(Z_{\tau^{\varepsilon,\varkappa}_n} - Z_{\tau^{\varkappa}_{n-1}})^2(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&\quad \times \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1} - \tau^{\varkappa}_{n-2})} \right\} \left\{ \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})} \right\} \chi_{(\tau^{\varepsilon,\varkappa}_n, \tau^{\varkappa}_n]}(u)
\end{aligned}$$

and

$$\begin{aligned}
v^{\varepsilon,\varkappa}_{n,4}(u)v^{\varepsilon,\varkappa}_{n,5}(u) &= 9(\tau^{\varepsilon,\varkappa}_n - u)(Z_u - Z_{\tau^{\varkappa}_{n-1}})^2(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})^2 \\
&\quad \times \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} \right\} \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varepsilon,\varkappa}_n]}(u) \\
v^{\varepsilon,\varkappa}_{n,4}(u)v^{\varepsilon,\varkappa}_{n,6}(u) &= -3(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)(Z_u - Z_{\tau^{\varkappa}_{n-1}})^2(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})^2 \\
&\quad \times \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})} \right\} \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varepsilon,\varkappa}_n]}(u) \\
v^{\varepsilon,\varkappa}_{n,4}(u)v^{\varepsilon,\varkappa}_{n,7}(u) &= -3(Z_u - Z_{\tau^{\varkappa}_{n-1}})^2(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})^2(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&\quad \times \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_n - \tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_n - \tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1} - \tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varepsilon,\varkappa}_n]}(u) \\
v^{\varepsilon,\varkappa}_{n,5}(u)v^{\varepsilon,\varkappa}_{n,6}(u) &= -3(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_n)(\tau^{\varepsilon,\varkappa}_n - u)(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})^2
\end{aligned}$$

$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^\varkappa - \tau_{n-1}^\varkappa)} \right\} \chi_{(\tau_{n-1}^\varkappa, \tau_n^{\varepsilon,\varkappa}]}(u)$$

$$v_{n,5}^{\varepsilon,\varkappa}(u) v_{n,7}^{\varepsilon,\varkappa}(u) = -3(\tau_n^{\varepsilon,\varkappa} - u)(Z_{\tau_{n-1}^\varkappa} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^2 (Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^2$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)} \right\} \chi_{(\tau_{n-1}^\varkappa, \tau_n^{\varepsilon,\varkappa}]}(u)$$

$$v_{n,6}^{\varepsilon,\varkappa}(u) v_{n,7}^{\varepsilon,\varkappa}(u) = (\tau_{n+1}^\varkappa - \tau_n^\varkappa)(Z_{\tau_{n-1}^\varkappa} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^2 (Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^2$$
$$\times \left\{ \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^\varkappa - \tau_{n-1}^\varkappa)} \right\} \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)} \right\} \chi_{(\tau_{n-1}^\varkappa, \tau_n^{\varepsilon,\varkappa}]}(u).$$

Again using (6), we see that

$$v_{n,1}^{\varepsilon,\varkappa}(u) v_{n,2}^{\varepsilon,\varkappa}(u) \approx -3(\tau_n^{\varepsilon,\varkappa} - \tau_{n-1}^\varkappa)^2 (\tau_{n-1}^{\varepsilon,\varkappa} - \tau_{n-2}^{\varepsilon,\varkappa})$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)} \right\} \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^\varkappa]}(u)$$

$$v_{n,1}^{\varepsilon,\varkappa}(u) v_{n,3}^{\varepsilon,\varkappa}(u) \approx -3(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^{\varepsilon,\varkappa} - \tau_{n-1}^\varkappa)^2$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^\varkappa - \tau_{n-1}^\varkappa)} \right\} \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^\varkappa]}(u)$$

$$v_{n,2}^{\varepsilon,\varkappa}(u) v_{n,3}^{\varepsilon,\varkappa}(u) \approx (\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^{\varepsilon,\varkappa} - \tau_{n-1}^\varkappa)(\tau_{n-1}^{\varepsilon,\varkappa} - \tau_{n-2}^{\varepsilon,\varkappa})$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)} \right\} \left\{ \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^\varkappa - \tau_{n-1}^\varkappa)} \right\} \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^\varkappa]}(u)$$

and

$$v_{n,4}^{\varepsilon,\varkappa}(u) v_{n,5}^{\varepsilon,\varkappa}(u) \approx 9(\tau_n^{\varepsilon,\varkappa} - u)(u - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-1}^{\varepsilon,\varkappa})$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \chi_{(\tau_{n-1}^\varkappa, \tau_n^{\varepsilon,\varkappa}]}(u)$$

$$v_{n,4}^{\varepsilon,\varkappa}(u) v_{n,6}^{\varepsilon,\varkappa}(u) \approx -3(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(u - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-1}^{\varepsilon,\varkappa})$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^\varkappa - \tau_{n-1}^\varkappa)} \right\} \chi_{(\tau_{n-1}^\varkappa, \tau_n^{\varepsilon,\varkappa}]}(u)$$

$$v_{n,4}^{\varepsilon,\varkappa}(u) v_{n,7}^{\varepsilon,\varkappa}(u) \approx -3(u - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-1}^{\varepsilon,\varkappa})(\tau_{n-1}^{\varepsilon,\varkappa} - \tau_{n-2}^{\varepsilon,\varkappa})$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)} \right\} \chi_{(\tau_{n-1}^\varkappa, \tau_n^{\varepsilon,\varkappa}]}(u)$$

$$v_{n,5}^{\varepsilon,\varkappa}(u) v_{n,6}^{\varepsilon,\varkappa}(u) \approx -3(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^{\varepsilon,\varkappa} - u)(\tau_{n-1}^\varkappa - \tau_{n-1}^{\varepsilon,\varkappa})$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^\varkappa - \tau_{n-1}^\varkappa)} \right\} \chi_{(\tau_{n-1}^\varkappa, \tau_n^{\varepsilon,\varkappa}]}(u)$$

$$v_{n,5}^{\varepsilon,\varkappa}(u) v_{n,7}^{\varepsilon,\varkappa}(u) \approx -3(\tau_n^{\varepsilon,\varkappa} - u)(\tau_{n-1}^\varkappa - \tau_{n-1}^{\varepsilon,\varkappa})(\tau_{n-1}^{\varepsilon,\varkappa} - \tau_{n-2}^{\varepsilon,\varkappa})$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)} \right\} \chi_{(\tau_{n-1}^\varkappa, \tau_n^{\varepsilon,\varkappa}]}(u)$$

$$v_{n,6}^{\varepsilon,\varkappa}(u) v_{n,7}^{\varepsilon,\varkappa}(u) \approx (\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-1}^{\varepsilon,\varkappa})(\tau_{n-1}^{\varepsilon,\varkappa} - \tau_{n-2}^{\varepsilon,\varkappa})$$
$$\times \left\{ \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^\varkappa - \tau_{n-1}^\varkappa)} \right\} \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)} \right\} \chi_{(\tau_{n-1}^\varkappa, \tau_n^{\varepsilon,\varkappa}]}(u).$$

We also have that

$$v_{n,1}^{\varepsilon,\varkappa}(u) v_{n,8}^{\varepsilon,\varkappa}(u) = -2(Z_u - Z_{\tau_{n-1}^\varkappa})(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau_{n-1}^\varkappa})^3 (Z_{\tau_{n-1}^\varkappa} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^\varkappa})$$
$$\times \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} \right\} \left\{ \frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)} \right\} \chi_{(\tau_n^{\varepsilon,\varkappa}, \tau_n^\varkappa]}(u)$$

$$
\begin{aligned}
v^{\varepsilon,\varkappa}_{n,1}(u)v^{\varepsilon,\varkappa}_{n,9}(u) &= 2(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n}} - Z_{\tau^{\varkappa}_{n-1}})^3(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}})(Z_{\tau^{\varkappa}_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}}) \\
&\qquad \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varepsilon,\varkappa}_{n},\tau^{\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,2}(u)v^{\varepsilon,\varkappa}_{n,8}(u) &= 2(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n}} - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varepsilon,\varkappa}_{n},\tau^{\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,2}(u)v^{\varepsilon,\varkappa}_{n,9}(u) &= -2(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n}} - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}})(Z_{\tau^{\varkappa}_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varepsilon,\varkappa}_{n},\tau^{\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,3}(u)v^{\varepsilon,\varkappa}_{n,8}(u) &= 2(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n})(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n}} - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n})(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varepsilon,\varkappa}_{n},\tau^{\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,3}(u)v^{\varepsilon,\varkappa}_{n,9}(u) &= -2(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n})(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n}} - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}})(Z_{\tau^{\varkappa}_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n})(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varepsilon,\varkappa}_{n},\tau^{\varkappa}_{n}]}(u)
\end{aligned}
$$

and

$$
\begin{aligned}
v^{\varepsilon,\varkappa}_{n,4}(u)v^{\varepsilon,\varkappa}_{n,8}(u) &= 6(Z_u - Z_{\tau^{\varkappa}_{n-1}})^3(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})^2(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varkappa}_{n-1},\tau^{\varepsilon,\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,4}(u)v^{\varepsilon,\varkappa}_{n,9}(u) &= -6(Z_u - Z_{\tau^{\varkappa}_{n-1}})^3(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}})(Z_{\tau^{\varkappa}_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varkappa}_{n-1},\tau^{\varepsilon,\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,5}(u)v^{\varepsilon,\varkappa}_{n,8}(u) &= 6(\tau^{\varepsilon,\varkappa}_{n} - u)(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})^2(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varkappa}_{n-1},\tau^{\varepsilon,\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,5}(u)v^{\varepsilon,\varkappa}_{n,9}(u) &= -6(\tau^{\varepsilon,\varkappa}_{n} - u)(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}})(Z_{\tau^{\varkappa}_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} + \frac{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})^2} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varkappa}_{n-1},\tau^{\varepsilon,\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,6}(u)v^{\varepsilon,\varkappa}_{n,8}(u) &= -2(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n})(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})^2(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n})(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varkappa}_{n-1},\tau^{\varepsilon,\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,6}(u)v^{\varepsilon,\varkappa}_{n,9}(u) &= 2(\tau^{\varkappa}_{n+1} - \tau^{\varkappa}_{n})(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}})(Z_{\tau^{\varkappa}_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n-1}}{(\tau^{\varkappa}_{n+1}-\tau^{\varkappa}_{n})(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varkappa}_{n-1},\tau^{\varepsilon,\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,7}(u)v^{\varepsilon,\varkappa}_{n,8}(u) &= -2(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varkappa}_{n-1},\tau^{\varepsilon,\varkappa}_{n}]}(u) \\
v^{\varepsilon,\varkappa}_{n,7}(u)v^{\varepsilon,\varkappa}_{n,9}(u) &= 2(Z_u - Z_{\tau^{\varkappa}_{n-1}})(Z_{\tau^{\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varkappa}_{n-2}})(Z_{\tau^{\varkappa}_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}}) \\
&\qquad \times \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \left\{ \frac{\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-2}}{(\tau^{\varkappa}_{n}-\tau^{\varkappa}_{n-1})(\tau^{\varkappa}_{n-1}-\tau^{\varkappa}_{n-2})} \right\} \chi_{(\tau^{\varkappa}_{n-1},\tau^{\varepsilon,\varkappa}_{n}]}(u)
\end{aligned}
$$

and finally

$$v^{\varepsilon,\varkappa}_{n,8}(u)v^{\varepsilon,\varkappa}_{n,9}(u) = -4(Z_u - Z_{\tau^\varkappa_{n-1}})^2(Z_{\tau^\varkappa_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})^2(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})$$
$$\times \left\{\frac{\tau^\varkappa_n - \tau^\varkappa_{n-2}}{(\tau^\varkappa_n - \tau^\varkappa_{n-1})(\tau^\varkappa_{n-1} - \tau^\varkappa_{n-2})}\right\}\left\{\frac{\tau^\varkappa_n - \tau^\varkappa_{n-2}}{(\tau^\varkappa_n - \tau^\varkappa_{n-1})(\tau^\varkappa_{n-1} - \tau^\varkappa_{n-2})}\right\}$$

4.4. **Simplifications: time scales.** Let's next use the actual sizes of the increments. If $\varepsilon \ll \varkappa$, we have that

$$\tau^{\varepsilon,\varkappa}_n - \tau^\varkappa_{n-1} \approx \tau^\varkappa_n - \tau^\varkappa_{n-1} = \varkappa$$
$$\tau^\varkappa_n - \tau^{\varepsilon,\varkappa}_n = \varepsilon\lambda.$$

As a result,

$$(v^{\varepsilon,\varkappa}_{n,1}(u))^2 \approx 15\varkappa^3\left\{\frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2}\right\}^2 \chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u) = 240\varkappa\chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u)$$
$$(v^{\varepsilon,\varkappa}_{n,2}(u))^2 \approx 3\varkappa^3\left\{\frac{2\varkappa}{\varkappa^2}\right\}^2 \chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u) = 12\varkappa\chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u)$$
$$(v^{\varepsilon,\varkappa}_{n,3}(u))^2 \approx \varkappa^3\left\{\frac{2\varkappa}{\varkappa^2}\right\}^2 \chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u) = 4\varkappa\chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u)$$

and

$$(v^{\varepsilon,\varkappa}_{n,4}(u))^2 \approx 27\varepsilon\lambda\left(u - \tau^\varkappa_{n-1}\right)^2\left\{\frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2}\right\}^2 \chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u) = 432\varepsilon\lambda\left(\frac{u - \tau^\varkappa_{n-1}}{\varkappa}\right)^2 \chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u)$$
$$(v^{\varepsilon,\varkappa}_{n,5}(u))^2 \approx 9\varepsilon\lambda\left(\tau^\varkappa_n - u\right)^2\left\{\frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2}\right\}^2 \chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u) = 144\varepsilon\lambda\left(\frac{\tau^\varkappa_n - u}{\varkappa}\right)^2 \chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u)$$
$$(v^{\varepsilon,\varkappa}_{n,6}(u))^2 \approx \varepsilon\lambda\varkappa^2\left\{\frac{2\varkappa}{\varkappa^2}\right\}^2 \chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u) = 4\varepsilon\lambda\chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u)$$
$$(v^{\varepsilon,\varkappa}_{n,7}(u))^2 \approx 3\varepsilon\lambda\varkappa^2\left\{\frac{2\varkappa}{\varkappa^2}\right\}^2 \chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u) = 12\varepsilon\lambda\chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u)$$

and

$$(v^{\varepsilon,\varkappa}_{n,8}(u))^2 \approx 4\varkappa\varepsilon\lambda\left(u - \tau^\varkappa_{n-1}\right)\left\{\frac{2\varkappa}{\varkappa^2}\right\}^2 = 16\varkappa\varepsilon\lambda\left(\frac{u - \tau^\varkappa_{n-1}}{\varkappa}\right)$$
$$(v^{\varepsilon,\varkappa}_{n,9}(u))^2 \approx 4\varkappa\varepsilon\lambda\left(u - \tau^\varkappa_{n-1}\right)\varkappa\left\{\frac{2\varkappa}{\varkappa^2}\right\}^2 = 16\varkappa\varepsilon\lambda\left(\frac{u - \tau^\varkappa_{n-1}}{\varkappa}\right)$$

We also have that

$$v^{\varepsilon,\varkappa}_{n,1}(u)v^{\varepsilon,\varkappa}_{n,2}(u) \approx -3\varkappa^3\left\{\frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2}\right\}\left\{\frac{2\varkappa}{\varkappa^2}\right\}\chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u) = -24\varkappa\chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u)$$
$$v^{\varepsilon,\varkappa}_{n,1}(u)v^{\varepsilon,\varkappa}_{n,3}(u) \approx -3\varkappa^3\left\{\frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2}\right\}\left\{\frac{2\varkappa}{\varkappa^2}\right\}\chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u) = -24\varkappa\chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u)$$
$$v^{\varepsilon,\varkappa}_{n,2}(u)v^{\varepsilon,\varkappa}_{n,3}(u) \approx \varkappa^3\left\{\frac{2\varkappa}{\varkappa^2}\right\}\left\{\frac{2\varkappa}{\varkappa^2}\right\}\chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u) = 4\varkappa\chi_{(\tau^\varkappa_n - \varepsilon\lambda, \tau^\varkappa_n]}(u)$$

and

$$v^{\varepsilon,\varkappa}_{n,4}(u)v^{\varepsilon,\varkappa}_{n,5}(u) \approx 9\varepsilon\lambda\left(\tau^\varkappa_n - u\right)\left(u - \tau^\varkappa_{n-1}\right)\left\{\frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2}\right\}\left\{\frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2}\right\}\chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u)$$
$$= 144\varepsilon\lambda\left(\frac{\tau^\varkappa_n - u}{\varkappa}\right)\left(\frac{u - \tau^\varkappa_{n-1}}{\varkappa}\right)\chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u)$$
$$v^{\varepsilon,\varkappa}_{n,4}(u)v^{\varepsilon,\varkappa}_{n,6}(u) \approx -3\varepsilon\lambda\varkappa\left(u - \tau^\varkappa_{n-1}\right)\left\{\frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2}\right\}\left\{\frac{2\varkappa}{\varkappa^2}\right\}\chi_{(\tau^\varkappa_{n-1}, \tau^\varkappa_n]}(u)$$

$$= -24\varepsilon\lambda \left(\frac{u - \tau^{\varkappa}_{n-1}}{\varkappa} \right) \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varkappa}_n]}(u)$$

$$v^{\varepsilon,\varkappa}_{n,4}(u) v^{\varepsilon,\varkappa}_{n,7}(u) \approx -3\varepsilon\lambda\varkappa \left(u - \tau^{\varkappa}_{n-1} \right) \left\{ \frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2} \right\} \left\{ \frac{2\varkappa}{\varkappa^2} \right\} \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varkappa}_n]}(u)$$

$$= -24\varepsilon\lambda\varkappa \left(\frac{u - \tau^{\varkappa}_{n-1}}{\varkappa} \right) \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varkappa}_n]}(u)$$

$$v^{\varepsilon,\varkappa}_{n,5}(u) v^{\varepsilon,\varkappa}_{n,6}(u) \approx -3\varepsilon\lambda\varkappa \left(\tau^{\varkappa}_n - u \right) \left\{ \frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2} \right\} \left\{ \frac{2\varkappa}{\varkappa^2} \right\} \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varkappa}_n]}(u)$$

$$= -24\varepsilon\lambda \left(\frac{\tau^{\varkappa}_n - u}{\varkappa} \right) \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varkappa}_n]}(u)$$

$$v^{\varepsilon,\varkappa}_{n,5}(u) v^{\varepsilon,\varkappa}_{n,7}(u) \approx -3\varepsilon\lambda\varkappa \left(\tau^{\varkappa}_n - u \right) \left\{ \frac{2\varkappa}{\varkappa^2} + \frac{2\varkappa}{\varkappa^2} \right\} \left\{ \frac{2\varkappa}{\varkappa^2} \right\} \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varkappa}_n]}(u)$$

$$= -24\varepsilon\lambda \left(\frac{\tau^{\varkappa}_n - u}{\varkappa} \right) \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varkappa}_n]}(u)$$

$$v^{\varepsilon,\varkappa}_{n,6}(u) v^{\varepsilon,\varkappa}_{n,7}(u) \approx \varepsilon\lambda\varkappa^2 \left\{ \frac{2\varkappa}{\varkappa^2} \right\} \left\{ \frac{2\varkappa}{\varkappa^2} \right\} \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varkappa}_n]}(u)$$

$$= 4\varepsilon\lambda \chi_{(\tau^{\varkappa}_{n-1}, \tau^{\varkappa}_n]}(u)$$

Summing things up, we have that We have that

$$\int_{u=\tau^{\varkappa}_{n-1}}^{\tau^{\varkappa}_n} v^{\varepsilon,\varkappa}_{n,j}(u) v^{\varepsilon,\varkappa}_{n,j'}(u) du \approx \varepsilon\lambda C_{j,j'}\varkappa$$

where

$$C_{1,1} = 240$$
$$C_{2,2} = 12$$
$$C_{3,3} = 4$$

and

$$C_{4,4} = 144$$
$$C_{5,5} = 48$$
$$C_{6,6} = 4$$
$$C_{7,7} = 12$$

and

$$C_{8,8} = 8$$
$$C_{9,9} = 8$$

and

$$C_{1,2} = -24$$
$$Ci_{1,3} = -24$$
$$Ci_{2,3} = 4$$

and

$$C_{4,5} = 24$$
$$C_{4,6} = -12$$
$$C_{4,7} = -12$$
$$C_{5,6} = -12$$
$$Ci_{5,7} = -12$$

$$C_{6,7} = 4$$

The sum of all of these constants is

$$C_{(8)} \stackrel{\text{def}}{=} 416. \tag{8}$$

4.5. **Error terms.** Let's finally consider the remaining terms, which are error terms. Define the index set

$$\mathcal{J} \stackrel{\text{def}}{=} \{(1,8),(1,9),(2,8),(3,8),(3,9),(4,8),(4,9),(5,8),(5,9),(6,8),(6,9),(7,8),(7,9),(8,9)\}.$$

We want to show that

$$\mathbb{E}[A_t^{\varepsilon,\varkappa,j,j'}] \ll \varepsilon\lambda \tag{9}$$

for $(j,j') \in \mathcal{J}$.

To proceed, we use the fact that the increments of Z are independent. Thus

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,1}^{\varepsilon,\varkappa}(u) v_{n,8}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,1}^{\varepsilon,\varkappa}(u) v_{n,9}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-2}^{\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,2}^{\varepsilon,\varkappa}(u) v_{n,8}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-1}^{\varkappa}} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,3}^{\varepsilon,\varkappa}(u) v_{n,8}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,3}^{\varepsilon,\varkappa}(u) v_{n,9}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-2}^{\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}}\right] = 0$$

and

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,4}^{\varepsilon,\varkappa}(u) v_{n,8}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,4}^{\varepsilon,\varkappa}(u) v_{n,9}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-2}^{\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}}\right] = 0)$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,5}^{\varepsilon,\varkappa}(u) v_{n,8}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,5}^{\varepsilon,\varkappa}(u) v_{n,9}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-2}^{\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,6}^{\varepsilon,\varkappa}(u) v_{n,8}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,6}^{\varepsilon,\varkappa}(u) v_{n,9}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-2}^{\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,7}^{\varepsilon,\varkappa}(u) v_{n,8}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varkappa}}\right] = 0$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^{\varkappa}}^{\tau_n^{\varkappa}} v_{n,7}^{\varepsilon,\varkappa}(u) v_{n,9}^{\varepsilon,\varkappa}(u)du\right] = 0 \qquad \text{since } \mathbb{E}\left[Z_{\tau_{n-2}^{\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}}\right] = 0$$

and finally

$$\mathbb{E}\left[\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n} v^{\varepsilon,\varkappa}_{n,8}(u)v^{\varepsilon,\varkappa}_{n,9}(u)du\right]=0 \qquad \text{since } \mathbb{E}\left[Z_{\tau^\varkappa_{n-2}}-Z_{\tau^{\varepsilon,\varkappa}_{n-2}}\right]=0$$

We note that $v^{\varepsilon,\varkappa}_{n,j}$ is measurable

$$\sigma\{Z_s-Z_{\tau^{\varepsilon,\varkappa}_{n-2}};\ \tau^{\varepsilon,\varkappa}_{n-2}\le s\le \tau^\varkappa_{n+1}\}.$$

Thus for $(j,j')\in\mathscr{J}$, independence implies that

$$\mathbb{E}\left[\left\{\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n} v^{\varepsilon,\varkappa}_{n,j}(u)v^{\varepsilon,\varkappa}_{n,j'}(u)du\right\}\left\{\int_{u'=\tau^\varkappa_{n'-1}}^{\tau^\varkappa_{n'}} v^{\varepsilon,\varkappa}_{n',j}(u')v^{\varepsilon,\varkappa}_{n,j'}(u')du'\right\}\right]$$
$$\mathbb{E}\left[\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n} v^{\varepsilon,\varkappa}_{n,j}(u)v^{\varepsilon,\varkappa}_{n,j'}(u)du\right]\mathbb{E}\left[\int_{u'=\tau^\varkappa_{n'-1}}^{\tau^\varkappa_{n'}} v^{\varepsilon,\varkappa}_{n',j}(u')v^{\varepsilon,\varkappa}_{n,j'}(u')du\right]=0$$

if $|n-n'|>3$. If $|n-n'|\le 3$,

$$\begin{aligned}
&\mathbb{E}\left[\left\{\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n} v^{\varepsilon,\varkappa}_{n,j}(u)v^{\varepsilon,\varkappa}_{n,j'}(u)du\right\}\left\{\int_{u'=\tau^\varkappa_{n'-1}}^{\tau^\varkappa_{n'}} v^{\varepsilon,\varkappa}_{n',j}(u')v^{\varepsilon,\varkappa}_{n,j'}(u')du'\right\}\right]\\
&\le \mathbb{E}\left[\left\{\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n}\left(v^{\varepsilon,\varkappa}_{n,j}(u)\right)^2du\right\}^{1/2}\left\{\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n}\left(v^{\varepsilon,\varkappa}_{n,j'}(u)\right)^2du\right\}^{1/2}\right.\\
&\qquad\left.\times\left\{\int_{u'=\tau^\varkappa_{n'-1}}^{\tau^\varkappa_{n'}}\left(v^{\varepsilon,\varkappa}_{n',j}(u')\right)^2du\right\}^{1/2}\left\{\int_{u'=\tau^\varkappa_{n'-1}}^{\tau^\varkappa_{n'}}\left(v^{\varepsilon,\varkappa}_{n,j'}(u')\right)^2du'\right\}^{1/2}\right]\\
&\le \mathbb{E}\left[\left\{\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n}\left(v^{\varepsilon,\varkappa}_{n,j}(u)\right)^2du\right\}^2\right]^{1/4}\mathbb{E}\left[\left\{\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n}\left(v^{\varepsilon,\varkappa}_{n,j'}(u)\right)^2du\right\}^2\right]^{1/4}\\
&\qquad\times\mathbb{E}\left[\left\{\int_{u'=\tau^\varkappa_{n'-1}}^{\tau^\varkappa_{n'}}\left(v^{\varepsilon,\varkappa}_{n',j}(u')\right)^2du\right\}^2\right]^{1/4}\mathbb{E}\left[\left\{\int_{u'=\tau^\varkappa_{n'-1}}^{\tau^\varkappa_{n'}}\left(v^{\varepsilon,\varkappa}_{n,j'}(u')\right)^2du'\right\}^2\right]^{1/4}\\
&\le \varkappa\mathbb{E}\left[\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n}\left(v^{\varepsilon,\varkappa}_{n,j}(u)\right)^4du\right]^{1/4}\mathbb{E}\left[\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n}\left(v^{\varepsilon,\varkappa}_{n,j'}(u)\right)^4du\right]\\
&\qquad\times\mathbb{E}\left[\int_{u'=\tau^\varkappa_{n'-1}}^{\tau^\varkappa_{n'}}\left(v^{\varepsilon,\varkappa}_{n',j}(u')\right)^4du\right]^{1/4}\mathbb{E}\left[\int_{u'=\tau^\varkappa_{n'-1}}^{\tau^\varkappa_{n'}}\left(v^{\varepsilon,\varkappa}_{n,j'}(u')\right)^4du'\right]^{1/4}
\end{aligned}$$

Note that

$$\begin{aligned}
\left(v^{\varepsilon,\varkappa}_{n,1}(u)\right)^4 &\overset{\text{def}}{=} (Z_{\tau^{\varepsilon,\varkappa}_n}-Z_{\tau^\varkappa_{n-1}})^{12}\left\{\frac{\tau^\varkappa_n-\tau^\varkappa_{n-2}}{(\tau^\varkappa_n-\tau^\varkappa_{n-1})^2}+\frac{\tau^\varkappa_{n+1}-\tau^\varkappa_{n-1}}{(\tau^\varkappa_n-\tau^\varkappa_{n-1})^2}\right\}^4\chi_{(\tau^{\varepsilon,\varkappa}_n,\tau^\varkappa_n]}(u)\\
&\approx \frac{(12)!}{2^6 6!}\sigma^{12}\varkappa^6\left\{\frac{2}{\varkappa}+\frac{2}{\varkappa}\right\}^4\chi_{(\tau^{\varepsilon,\varkappa}_n,\tau^\varkappa_n]}(u)=4^4\frac{(12)!}{2^6 6!}\sigma^{12}\varkappa^2\chi_{(\tau^{\varepsilon,\varkappa}_n,\tau^\varkappa_n]}(u)\\
\left(v^{\varepsilon,\varkappa}_{n,2}(u)\right)^4 &\overset{\text{def}}{=} (Z_{\tau^{\varepsilon,\varkappa}_n}-Z_{\tau^\varkappa_{n-1}})^4(Z_{\tau^{\varepsilon,\varkappa}_{n-1}}-Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^8\left\{\frac{\tau^\varkappa_n-\tau^\varkappa_{n-2}}{(\tau^\varkappa_n-\tau^\varkappa_{n-1})(\tau^\varkappa_{n-1}-\tau^\varkappa_{n-2})}\right\}^4\chi_{(\tau^{\varepsilon,\varkappa}_n,\tau^\varkappa_n]}(u)\\
&\approx \frac{4!}{2^2 2!}\frac{8!}{2^4 4!}\sigma^{12}\varkappa^6\left\{\frac{2}{\varkappa}\right\}^4\chi_{(\tau^{\varepsilon,\varkappa}_n,\tau^\varkappa_n]}(u)=2^4\frac{4!}{2^2 2!}\frac{8!}{2^4 4!}\sigma^{12}\varkappa^2\chi_{(\tau^{\varepsilon,\varkappa}_n,\tau^\varkappa_n]}(u)\\
\left(v^{\varepsilon,\varkappa}_{n,3}(u)\right)^4 &\overset{\text{def}}{=} (\tau^\varkappa_{n+1}-\tau^\varkappa_n)^4(Z_{\tau^{\varepsilon,\varkappa}_n}-Z_{\tau^\varkappa_{n-1}})^4\left\{\frac{\tau^\varkappa_{n+1}-\tau^\varkappa_{n-1}}{(\tau^\varkappa_{n+1}-\tau^\varkappa_n)(\tau^\varkappa_n-\tau^\varkappa_{n-1})}\right\}^4\varkappa^2\chi_{(\tau^{\varepsilon,\varkappa}_n,\tau^\varkappa_n]}(u)
\end{aligned}$$

$$\approx \frac{4!}{2^2 2!}\sigma^4\varkappa^6\left\{\frac{2}{\varkappa}\right\}^4\chi_{(\tau_n^{\varepsilon,\varkappa},\tau_n^\varkappa]}(u) = 2^4\frac{4!}{2^2 2!}\sigma^4\varkappa^2\chi_{(\tau_n^{\varepsilon,\varkappa},\tau_n^\varkappa]}(u)$$

$$\left(v_{n,4}^{\varepsilon,\varkappa}(u)\right)^4 \overset{\text{def}}{=} 81(Z_u - Z_{\tau_{n-1}^\varkappa})^8(Z_{\tau_{n-1}^\varkappa} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^4\left\{\frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2}\right\}^4\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$= 81\frac{8!}{2^4 4!}\frac{4!}{2^2 2!}\left(\frac{u-\tau_{n-1}^\varkappa}{\varkappa}\right)^8\sigma^{12}\varkappa^4(\varepsilon\lambda)^2\left\{\frac{4}{\varkappa}\right\}^4\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$= 81\cdot 4^4\cdot\frac{8!}{2^4 4!}\frac{4!}{2^2 2!}\left(\frac{u-\tau_{n-1}^\varkappa}{\varkappa}\right)^8\sigma^{12}(\varepsilon\lambda)^2\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$\left(v_{n,5}^{\varepsilon,\varkappa}(u)\right)^4 \overset{\text{def}}{=} 81(\tau_n^{\varepsilon,\varkappa} - u)^4(Z_{\tau_{n-1}^\varkappa} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^4\left\{\frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2} + \frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)^2}\right\}^4\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$= 81\frac{4!}{2^2 2!}\left(\frac{\tau_n^\varkappa - u}{\varkappa}\right)^4\varkappa^4(\varepsilon\lambda)^2\sigma^4\left\{\frac{4}{\varkappa}\right\}^4\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$= 81\cdot 4^4\cdot\frac{4!}{2^2 2!}\left(\frac{\tau_n^\varkappa - u}{\varkappa}\right)^4\sigma^4(\varepsilon\lambda)^2\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$\left(v_{n,6}^{\varepsilon,\varkappa}(u)\right)^4 \overset{\text{def}}{=} (\tau_{n+1}^\varkappa - \tau_n^\varkappa)^4(Z_{\tau_{n-1}^\varkappa} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^4\left\{\frac{\tau_{n+1}^\varkappa - \tau_{n-1}^\varkappa}{(\tau_{n+1}^\varkappa - \tau_n^\varkappa)(\tau_n^\varkappa - \tau_{n-1}^\varkappa)}\right\}^4\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$= \frac{4!}{2^2 2!}\varkappa^4(\varepsilon\lambda)^2\sigma^4\left\{\frac{2}{\varkappa}\right\}^4\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u) = 2^4\frac{4!}{2^2 2!}\sigma^4(\varepsilon\lambda)^2\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$\left(v_{n,7}^{\varepsilon,\varkappa}(u)\right)^4 \overset{\text{def}}{=} (Z_{\tau_{n-1}^\varkappa} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^4(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^8\left\{\frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)}\right\}^4\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$= \frac{4!}{2^2 2!}\frac{8!}{2^4 4!}\sigma^{12}\varkappa^4(\varepsilon\lambda)^2\left\{\frac{2}{\varkappa}\right\}^4\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u) = 2^4\frac{4!}{2^2 2!}\frac{8!}{2^4 4!}\sigma^{12}(\varepsilon\lambda)^2\chi_{(\tau_{n-1}^\varkappa,\tau_n^{\varepsilon,\varkappa}]}(u)$$

$$\left(v_{n,8}^{\varepsilon,\varkappa}(u)\right)^4 \overset{\text{def}}{=} 16(Z_u - Z_{\tau_{n-1}^\varkappa})^4(Z_{\tau_{n-1}^\varkappa} - Z_{\tau_{n-1}^{\varepsilon,\varkappa}})^4(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^\varkappa})^4\left\{\frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)}\right\}^4$$

$$= 16\left(\frac{4!}{2^2 2!}\right)^3\left(\frac{u-\tau_{n-1}^\varkappa}{\varkappa}\right)^2\sigma^{12}\varkappa^4(\varepsilon\lambda)^2\left\{\frac{2}{\varkappa}\right\}^4 = 16\cdot 2^4\left(\frac{4!}{2^2 2!}\right)^3\left(\frac{u-\tau_{n-1}^\varkappa}{\varkappa}\right)^2\sigma^{12}(\varepsilon\lambda)^2$$

$$\left(v_{n,9}^{\varepsilon,\varkappa}(u)\right)^4 \overset{\text{def}}{=} 16(Z_u - Z_{\tau_{n-1}^\varkappa})^4(Z_{\tau_{n-1}^{\varepsilon,\varkappa}} - Z_{\tau_{n-2}^\varkappa})^4(Z_{\tau_{n-2}^\varkappa} - Z_{\tau_{n-2}^{\varepsilon,\varkappa}})^4\left\{\frac{\tau_n^\varkappa - \tau_{n-2}^\varkappa}{(\tau_n^\varkappa - \tau_{n-1}^\varkappa)(\tau_{n-1}^\varkappa - \tau_{n-2}^\varkappa)}\right\}^4$$

$$= 16\left(\frac{4!}{2^2 2!}\right)^3\left(\frac{u-\tau_{n-1}^\varkappa}{\varkappa}\right)^2\sigma^{12}\varkappa^4(\varepsilon\lambda)^2\left\{\frac{2}{\varkappa}\right\}^4 = 16\cdot 2^4\left(\frac{4!}{2^2 2!}\right)^3\left(\frac{u-\tau_{n-1}^\varkappa}{\varkappa}\right)^2\sigma^{12}(\varepsilon\lambda)^2$$

Thus there is a constant $\mathbf{K} > 0$ such that

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^\varkappa}^{\tau_n^\varkappa}\left(v_{n,j}^{\varepsilon,\varkappa}(u)\right)^4 du\right] \le \mathbf{K}\varkappa^2\varepsilon\lambda \qquad 1 \le j \le 3$$

$$\mathbb{E}\left[\int_{u=\tau_{n-1}^\varkappa}^{\tau_n^\varkappa}\left(v_{n,j}^{\varepsilon,\varkappa}(u)\right)^4 du\right] \le \mathbf{K}\left(\varepsilon\lambda\right)^2\varkappa \qquad 4 \le j \le 9$$

Note that if $\varepsilon \ll \varkappa$, then $\varepsilon^2\varkappa \ll \varepsilon\varkappa^2$. Also, if $(j,j') \in \mathcal{J}$, then either j or j', but not both, can be in $\{1,2,3\}$. Collecting things together, there is a $\mathbf{K}' > 0$ such that

$$\mathbb{E}\left[\left|A_t^{\varepsilon,\varkappa,j,j}\right|^2\right] \le \sum_{\substack{0\le n,n'\le\lfloor T/(\varkappa\alpha)\rfloor \\ |n-n'|\le 3}}\left\{(\mathbf{K}\varepsilon^2\varkappa)^{1/4}\right\}^2\left\{(\mathbf{K}\varepsilon\varkappa^2)^{1/4}\right\}^4$$

$$\leq \mathbf{K}' \frac{\varepsilon^{3/2}\varkappa^{3/2}}{\varkappa} = \mathbf{K}'\varepsilon^{3/2}\varkappa^{1/2} \ll \varepsilon$$

Collecting our calculations together thus far, we see that there is a constant $\mathbf{K} > 0$ such that

$$\mathbb{E}\left[\left|A_t^{\varepsilon,\varkappa,j,j}\right|^2\right] \leq \frac{\mathbf{K}}{\varkappa}\left(\sqrt{(\varkappa\varepsilon)^2}\right)^2 = \mathbf{K}\varepsilon^2\varkappa,$$

giving us (9).

The only remaining term is $v_{n,2}^{\varepsilon,\varkappa}(u)v_{n,9}^{\varepsilon,\varkappa}(u)$. We have that

$$\begin{aligned}
&(Z_u - Z_{\tau^\varkappa_{n-1}})(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau^\varkappa_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&= (Z_u - Z_{\tau_n^{\varepsilon,\varkappa}})(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau^\varkappa_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&\qquad + (Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau^\varkappa_{n-1}})^2(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&= (Z_u - Z_{\tau_n^{\varepsilon,\varkappa}})(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau^\varkappa_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&\qquad + (Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau^\varkappa_{n-1}})^2(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}} + Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&= 2(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau^\varkappa_{n-1}})^2(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})^2(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&\qquad + (Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau^\varkappa_{n-1}})^2\left\{(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})^3(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}}) + (Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^3\right\} \\
&\qquad + (Z_u - Z_{\tau_n^{\varepsilon,\varkappa}})(Z_{\tau_n^{\varepsilon,\varkappa}} - Z_{\tau^\varkappa_{n-1}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}})(Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})(Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}})^2 \\
&\approx 2(\tau_n^{\varepsilon,\varkappa} - \tau^\varkappa_{n-1})(\tau^{\varepsilon,\varkappa}_{n-1} - \tau^\varkappa_{n-2})(\tau^\varkappa_{n-2} - \tau^{\varepsilon,\varkappa}_{n-2})
\end{aligned}$$

(we have used here the calculation that $xy(x+y)^2 = 2x^2y^2 + x^3y + xy^3$, with $x = Z_{\tau^{\varepsilon,\varkappa}_{n-1}} - Z_{\tau^\varkappa_{n-2}}$ and $y = Z_{\tau^\varkappa_{n-2}} - Z_{\tau^{\varepsilon,\varkappa}_{n-2}}$). Thus

$$\begin{aligned}
v_{n,2}^{\varepsilon,\varkappa}(u)v_{n,9}^{\varepsilon,\varkappa}(u) &\approx 2(\tau_n^{\varepsilon,\varkappa} - \tau^\varkappa_{n-1})(\tau^{\varepsilon,\varkappa}_{n-1} - \tau^\varkappa_{n-2})(\tau^\varkappa_{n-2} - \tau^{\varepsilon,\varkappa}_{n-2}) \\
&\qquad \times \left\{\frac{\tau^\varkappa_n - \tau^\varkappa_{n-2}}{(\tau^\varkappa_n - \tau^\varkappa_{n-1})(\tau^\varkappa_{n-1} - \tau^\varkappa_{n-2})}\right\}\left\{\frac{\tau^\varkappa_n - \tau^\varkappa_{n-2}}{(\tau^\varkappa_n - \tau^\varkappa_{n-1})(\tau^\varkappa_{n-1} - \tau^\varkappa_{n-2})}\right\}\chi_{(\tau_n^{\varepsilon,\varkappa},\tau_n^\varkappa]}(u) \\
&\approx 2\varkappa^2\varepsilon\lambda\left\{\frac{2\varkappa}{\varkappa^2}\right\}\left\{\frac{2\varkappa}{\varkappa^2}\right\}\chi_{(\tau_n^{\varepsilon,\varkappa},\tau_n^\varkappa]}(u) \\
&\approx 8\varepsilon\lambda\chi_{(\tau_n^{\varepsilon,\varkappa},\tau_n^\varkappa]}(u)
\end{aligned}$$

Thus

$$\int_{u=\tau^\varkappa_{n-1}}^{\tau^\varkappa_n} v_{n,2}^{\varepsilon,\varkappa}(u)v_{n,9}^{\varepsilon,\varkappa}(u)du \approx 8(\varepsilon\lambda)^2\varkappa;$$

this term is thus also negligible.

Summing up, we have the proof of Theorem 4.1.

5. Stochastic Latency

As an extension, let's now allow latency to evolve, but on a time scale which is of order 1. Let λ be a standard CIR process, uncorrelated with W; i.e.,

$$\begin{aligned}
d\lambda_t &= -\alpha(\lambda_t - \bar\lambda)dt + \sigma\sqrt{\lambda_t}dV_t \\
\lambda_0 &= \lambda_\circ
\end{aligned} \tag{10}$$

In fact, we could use any nonnegative and continuous process which is independent of W. and $\mathbb{P}$-a.s. bounded process. We now define

$$S_t^\varepsilon \overset{\text{def}}{=} S_{t-\varepsilon\lambda_t}.$$

Figure 2 shows what we are interested in.

In figure 2, the blue process is a realization of the 'fundamental' price, and the black process is the latency process.

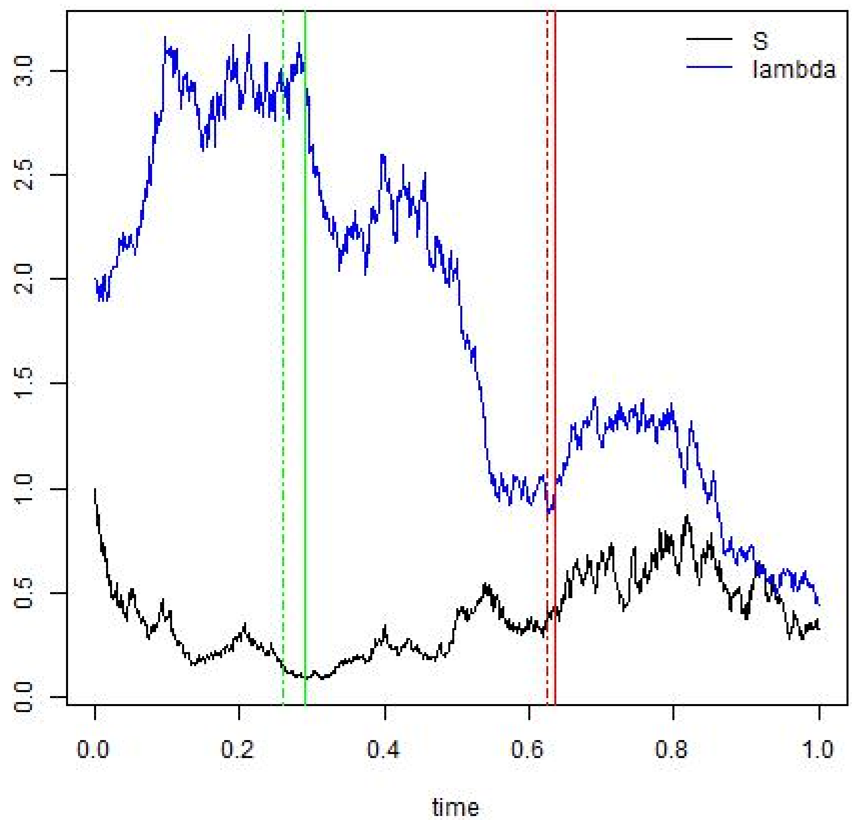

FIGURE 2. Stochastic Latency and Asset Price

The red and green solid vertical lines are trade times; the dotted lines are the times at which the fundamental prices is evaluated for the two trade times. For the red pair, the latency is smaller than for the green pair; thus the dotted lines are closer together for the red pair than the green pair, indicating that the price for the green pair is lagged less (and thus closer to the current true price) than for the red pair.

Corollary 5.1. *We have that*

$$\boldsymbol{VIV}^{\varepsilon,\varkappa}(T) \approx \boldsymbol{VIV}^{\varkappa}(T) + \varepsilon M_T$$

where M is a martingale with quadratic variation

$$\langle M \rangle_T = \int_{s=0}^{T} C_{(8)} \sigma^4 \lambda_s ds$$

where $C_{(8)}$ is the constant given in (8)*.*

Idea of Proof. The process λ of (10) fluctuations on time scales of order 1; since **VIV** is constructed from microscopic calculations, we can effectively compute **VIV** with mesoscopically (i.e., locally) constant latency, and then piece the result together.

A bit more precisely, fix $J\in\mathbb{N}$ with $J\gg 1$, and consider

$$\begin{aligned}&\mathbf{VIV}^{\varepsilon,\varkappa}(T)-\mathbf{VIV}^{\varkappa}(T)\\&\qquad=\sum_{j=1}^J\left\{\left\{\mathbf{VIV}^{\varepsilon,\varkappa}\left(\frac{j}{J}T\right)-\mathbf{VIV}^{\varkappa}\left(\frac{j}{J}T\right)\right\}-\left\{\mathbf{VIV}^{\varepsilon,\varkappa}\left(\frac{j-1}{J}T\right)-\mathbf{VIV}^{\varkappa}\left(\frac{j-1}{J}T\right)\right\}\right\}.\end{aligned}$$

By Theorem 4.1 , we have that

$$\begin{aligned}&\left\{\left\{\mathbf{VIV}^{\varepsilon,\varkappa}\left(\frac{j}{J}T\right)-\mathbf{VIV}^{\varkappa}\left(\frac{j}{J}T\right)\right\}-\left\{\mathbf{VIV}^{\varepsilon,\varkappa}\left(\frac{j-1}{J}T\right)-\mathbf{VIV}^{\varkappa}\left(\frac{j-1}{J}T\right)\right\}\right\}\\&\qquad\approx M^{(J)}_{jT/J}-M^{(J)}_{(j-1)T/J}\end{aligned}\tag{11}$$

where $M^{(J)}$ is a martingale with quadratic variation $\sigma^4\lambda((j-1)T/J)T/J$. The quadratic variation of the right-hand side of (11) is then

$$\sum_{j=1}^J C_{(8)}\sigma^4\lambda\left(\frac{j-1}{J}T\right)\frac{T}{J}\approx\int_{s=0}^T C_{(8)}\sigma^4\lambda(s)dsds.$$

with the last approximation holding as $J\nearrow\infty$. □

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